Dielectron widths of the S-, D-vector bottomonium states

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The dielectron widths of $\Upsilon(nS)$ $(n = 1, \ldots, 7)$ and vector decay constants are calculated using the Relativistic String Hamiltonian with a universal interaction. For $\Upsilon(nS)$ $(n = 1, 2, 3)$ the dielectron widths and their ratios are obtained in full agreement with the latest CLEO data. For $\Upsilon(10580)$ and $\Upsilon(11020)$ a good agreement with experiment is reached only if the $4S$—$3D$ mixing (with a mixing angle $\theta = 27^\circ \pm 4^\circ$) and $6S$—$5D$ mixing (with $\theta = 40^\circ \pm 5^\circ$) are taken into account. The possibility to observe higher "mixed D-wave" resonances, $\tilde{\Upsilon}(n\,^3D_1)$ with $n = 3, 4, 5$ is discussed. In particular, $\tilde{\Upsilon}(\approx 11120)$, originating from the pure $5\,^3D_1$ state, can acquire a rather large dielectron width, ~ 130 eV, so that this resonance may become manifest in the e^+e^- experiments. On the contrary, the widths of pure D-wave states are very small, $\Gamma_{ee}(n^3D_1) \leq 2$ eV.

1. INTRODUCTION

The spectrum of bottomonium is very rich with a large number of the levels below the $B\bar{B}$ threshold. Among them three well established $\Upsilon(n\,^3S_1)$ $(n = 1, 2, 3)$ mesons [1] and the $1\,^3D_2$ state, discovered in [2]. There exist numerous studies of low-lying bottomonium levels, where different QCD motivated models are used [3, 4, 5, 6, 7, 8, 9]. However, a number of these levels have not been observed yet.

Observation of the D-wave states which lie below the open beauty threshold, is a difficult experimental task, as demonstrated in the CLEO experiment [2], where to discover the $1\,^3D_2$ level, four-photon cascade measurements in the $\Upsilon(3S)$ radiative decays have been performed. In particular, neither the $1\,^3D_1$ state nor the members of the $2D$ multiplet, for which potential model calculations give masses around 10.45 GeV [4, 10], i.e., below the $B\bar{B}$ threshold, are as yet observed. One of the reasons for that is the very small dielectron widths of pure $n\,^3D_1$ states (for any n): here and in [10] their values ~ 1 eV are obtained. For that reason an observation of a pure D-wave vector state directly in e^+e^- experiments seems to be impossible at the present stage.

However, the situation may change for the D-wave vector states which lie above the open beauty threshold. For these bottomonium states the dielectron widths may become larger, as happens in the charmonium family, where the experimental dielectron width of $\psi(3770)$ (which is only 30 MeV above the $D\bar{D}$ threshold) is already ten times larger than for a pure $1\,^3D_1$ state [11, 12]. Moreover, the width of the $2\,^3D_1$ resonance $\psi(4160)$ is almost equal to that of $\psi(4040)$, which therefore cannot be considered as a pure $3\,^3S_1$ state. Such an increase of the dielectron width of a D-wave vector state and at the same time a decrease of the width of an S-wave state can occur if a rather large S—D mixing between both states takes place [11, 12, 13].

A theoretical study of the S—D mixing between vector states is more simple in bottomonium than in the charmonium family, since the experimental dielectron widths are now measured for the six states $\Upsilon(nS)$ $(n = 1, ..., 6)$ [1]. It is also essential that in the recent CLEO experiments the dielectron widths of low-lying levels, $\Upsilon(nS)$ $(n = 1, 2, 3)$, and their ratios were measured with great accuracy [14]. These three levels can indeed be considered as pure S-wave states, because for them the S—D mixing is possible only via tensor forces, which give very small mixing angle (see the Appendix). Then these pure S-wave states can be studied in the single-channel approach (SCA). Here in particular, we use the well-developed relativistic string Hamiltonian (RSH) [15]. Moreover, just for these levels a comparison of experimental and calculated dielectron widths and their ratios can be considered as an important test of the theoretical approach and also of the calculated wave functions (w.f.) at the origin.

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There are not many theoretical studies of higher bottomonium states [10, 16, 17, 18]. Strictly speaking, for this task one needs to solve a many-channel problem, knowing the interactions within all channels and between them. Unfortunately, this program is not realized now, although some important steps in this direction have been done recently in [19], where a theory of the interactions between the channels due to a strong coupling to virtual (open) $B\bar{B}$ ($B_s\bar{B}_s$) channels was developed.

In this paper, using the RSH for the low-lying states $\Upsilon(nS)(n = 1, 2, 3)$, we obtain a good description of the dielectron widths and their ratios. After that we apply the same approach to higher bottomonium states (above the $B\bar{B}$ threshold), where the accuracy of our calculations is becoming worse: in particular, within the SCA one cannot calculate the mass shift of a higher resonance, which can occur owing to coupling to open channel(s). However, for the dielectron widths it is most important to define the w.f. at the origin with a good accuracy.

There exist several arguments in favor of the validity of the the SCA for higher bottomonium states:

1. First, in charmonium this approximation gives the masses and the dielectron widths of $\psi(3770)$, $\psi(4040)$, $\psi(4160)$, and $\psi(4415)$ with a good accuracy, providing a self-consistent description of their dielectron widths [12, 13].

2. Secondly, an open channel (e.g. $B\bar{B}$) can be considered as a particular case of a four-quark system $Q\bar{Q}q\bar{q}$, and, as shown in [20], this open channel cannot significantly change the w.f. at the origin of heavy quarkonia, because the magnitude of a four-quark w.f. at the origin is two orders smaller than for a $Q\bar{Q}$ meson.

3. Finally, the masses of higher bottomonium states, calculated in SCA, appear to be rather close to the experimental values, giving a difference between them equal to at most 50 ± 15 MeV.

For pure $n\,^3D_1$ bottomonium states (any n) our calculations give very small dielectron widths, ≤ 2 eV, while for higher D-wave vector states their dielectron widths can increase owing to S—D mixing through an open channel(s). Here an important point is that the mass of a higher $n\,^3D_1$ state ($n \geq 3$) appears to be only 40-50 MeV larger than that of the $\Upsilon((n+1)S)$ state [4], [10], thus increasing the probability of the S—D mixing between these resonances. We will define the mixing angle here in a phenomenological way, as it was done in charmonium [11], [13].

Owing to the S—D mixing the dielectron widths of mixed D-wave bottomonium resonances appear to increase by two orders of magnitude and reach $\sim 100 \pm 30$ eV, while the dielectron widths of the initially pure $n\,^3S_1$ resonances ($n = 4, 5, 6$) decrease. We also calculate the vector decay constants in bottomonium and briefly discuss the possibility to observe "mixed D-wave" resonances in e^+e^- experiments.

2. MASS SPECTRUM

The spectrum and the w.f. of the bottomonium vector states ($L = 0, 2$) are calculated with the help of the RSH and a universal static potential from [21]. This Hamiltonian has been successfully applied to light [22] and heavy-light mesons [23, 24], and also to heavy quarkonia [7, 25, 26]. In bottomonium the spin-averaged masses $M(nL)$ of the nL multiplets are determined by a simpler mass formula than for other mesons, because it does not contain the self-energy and the string contributions which in bottomonium are very small, ≤ 1 MeV, and can be neglected [7]. As a result, the mass $M(nL)$ just coincides with the eigenvalue (e.v.) of the spinless Salpeter equation (SSE) or with the e.v. of the einbein equation, derived in the so-called einbein approximation (EA) [23].

Here we use the EA, because the nS-wave functions, defined by the EA equation, has an important advantage as compared to the solutions of SSE: they are finite near the origin, while the nS-w.f. of the SSE diverge for any n and have to be regularized (e.g. as in [4]), introducing unknown parameters. At the same time the difference between the EA and SSE masses is small, ≤ 15 MeV (the e.v. in the SSE are always smaller than the e.v. in the EA) and can be included in the theoretical error. In Table 1 the centroid masses $M(nL)$ are given both for the SSE and the EA, calculated for the same set of input parameters.

We do not discuss here the singlet ground state η_b, recently discovered by BaBar [27], because the hyperfine (HF) interaction introduces extra, not well-known parameters, while our goal here is to describe the bottomonium data, not introducing extra parameters and using a universal potential which contains only fundamental parameters - the QCD constant Λ and the string tension.

The masses of $\Upsilon(n\,^3S_1)$ are very close to the centroid masses $M(nS)$ (with the exception of the ground state), because the HF splittings are small [26]: for higher radial excitations the difference between $M(n\,^3S_1)$ and the centroid mass is ≤ 4 MeV for $n \geq 3$. Moreover, for the D-wave multiplets the fine-structure splittings are small [4] and therefore the calculated centroid masses $M(nD)$ coincide with $M(n\,^3D_1)$ within the theoretical error.

The RSH is defined by the expression from [15, 22]:

$$H_0 = \frac{\mathbf{p}^2 + m_b^2}{\omega} + \omega + V_B(r). \tag{1}$$

Here m_b is the pole mass of the b quark, for which the value $m_b = 4.832$ GeV is used. This number corresponds to the current mass $m_b = 4.235$ GeV, which coincides with the conventional current b-quark mass, equal to 4.20 ± 0.07 GeV [1]. The static potential $V_B(r)$, defined below in Eq.(5), contains the symbol B, which shows that this potential was derived in background perturbation theory [15].

In (1) the variable ω can be defined in two ways: If the extremum condition is put on the Hamiltonian H_0, ω is equal to the kinetic energy operator, $\omega = \sqrt{\mathbf{p}^2 + m_b^2}$. Substituting this operator ω into H_0, one arrives at the well-known SSE. However, the S-wave w.f. of the SSE diverge near the origin and for their definition one needs to use a regularization procedure, in this way introducing several additional parameters.

Instead we prefer to use the EA, where the variable ω is determined from another extremum condition, put on the e.v. $M(nL)$. Then $\omega(nL)$ is not an operator anymore, but is equal to the matrix element (m.e.) of the kinetic energy operator and plays the role of a dynamical (constituent) quark mass. This constituent mass $\omega(nL)$ grows with increasing quantum numbers and this fact appears to be very important for light and heavy-light mesons [23, 24], while in bottomonium the difference between the dynamical mass ω_{nL} and the pole mass m_b is not large, changing from ~ 170 MeV for the $1S$ ground state up to ~ 300 MeV for higher states, like $6S$.

In the framework of the EA the w.f. of heavy-light mesons have been calculated and successfully applied to determine the pseudoscalar decay constants of the D, D_s, B, and B_s mesons, giving a good agreement with experiment [24].

It is of interest to notice that in bottomonium the masses, calculated in EA and SSE, and also in the nonrelativistic (NR) case (where $\omega_{\rm NR}(nL) = m_b$ for all states), do not differ much, even for higher states: such mass differences are ≤ 40 MeV (see below and Tables 1 and 2). Still, the w.f. at the origin, calculated in EA, takes into account the relativistic corrections and gives rise to a better agreement with the experimental dielectron widths than in the NR approach.

In EA the masses $M(nL)$ are defined by the mass formula:

$$M(nL) = \omega_{nL} + \frac{m_b^2}{\omega_{nL}} + E_{nL}(\omega_{nL}), \tag{2}$$

where $\omega(nL)$ and the e.v. E_{nL} have to be defined solving two self-consistent equations [12, 23], namely

$$\left[\frac{\mathbf{p}^2}{\omega_{nL}} + V_B(r)\right] \varphi_{nL}(r) = E_{nL}\ \varphi_{nL}(r), \tag{3}$$

and the equation

$$\omega_{nL}^2 = m_b^2 - \frac{\partial E_{nL}}{\partial \omega_{nL}}. \tag{4}$$

In (3) we use for all mesons the universal static potential $V_B(r)$ from [7, 21]:

$$V_B(r) = \sigma(r)\, r - \frac{4}{3}\frac{\alpha_B(r)}{r}, \tag{5}$$

with the following set of the parameters:

$$\begin{array}{l} m_b = 4.832\,\mathrm{GeV},\ \ \Lambda_B(n_f = 5) = 0.335\ \mathrm{GeV}, \\ M_B = 0.95\,\mathrm{GeV},\ \ \sigma_0 = 0.178\ \mathrm{GeV}^2. \end{array} \tag{6}$$

The QCD (vector) constant Λ_B, which determines the vector coupling constant $\alpha_B(r)$ (see Eq. (7) below), depends on the number of flavors and can be expressed via the QCD constant in the $\overline{\rm MS}$ regularization scheme; the connection between both constants has been established in [21, 28]. In particular, the two-loop constant $\Lambda_B(n_f = 5) = 335$ MeV in Eq. (6) corresponds to the two-loop $\Lambda_{\overline{\rm MS}} = 244$ MeV, since they are related as $\Lambda_B(n_f = 5) = 1.3656\ \Lambda_{\overline{\rm MS}}$ [28]. However, one cannot exclude that for low-lying bottomonium levels, like $1S$, $1P$, and $1D$, the choice $n_f = 4$, equal to the number of active flavors, might be preferable, giving for their masses a better agreement with experiment. Here for simplicity we take $n_f = 5$ for all states, because we are mostly interested in higher states, above the open beauty threshold. The constant σ_0 occurs in the expression for the variable string tension $\sigma(r)$ given by Eq. (8).

The vector coupling in the coordinate space $\alpha_B(r)$ is defined via the strong coupling in the momentum space $\alpha_B(q)$ [21]:

$$\begin{aligned} \alpha_B(r) &= \frac{2}{\pi}\int\limits_0^\infty dq \frac{\sin(qr)}{q}\alpha_B(q), \\ \alpha_B(q) &= \frac{4\pi}{\beta_0 t_B}\left(1 - \frac{\beta_1}{\beta_0^2}\frac{\ln t_B}{t_B}\right) \end{aligned} \tag{7}$$

TABLE 1: The spin-averaged masses $M(nL)$ (MeV) of low-lying multiplets, calculated in nonrelativistic (NR) case, for spinless Salpeter equation (SSE), and in einbein approximation (EA). In all cases the parameters of $V_B(r)$ are taken from Eq. (6)

State	NR	SSE	EA	Exp. [1]
$1S$	9469	9453	9462	9460.30±0.26 (1^3S_1)
$1P$	9894	9884	9888	9900.1±0.6
$2S$	10028	10010	10021	10023.3±0.3 (2^3S_1)
$1D$	10153	10144	10146	10161.1±1.7 (1^2D_1)
$2P$	10270	10256	10261	10260.0±0.6
$1F$	10355	10345	10347	-
$3S$	10379	10356	10369	10355.2±0.5 (3^3S_1)
$2D$	10460	10446	10450	-
$3P$	10562	10541	10551	-

with $t_B = \frac{\ln(\mathbf{q}^2+M_B^2)}{\Lambda_B^2}$.

The solutions of Eq. (3) are calculated here considering two types of confining potential in (5): one with the string tension equal to a constant, $\sigma_0 = 0.178$ GeV^2, and the other with the string tension $\sigma(r)$ dependent on the $Q\bar{Q}$ separation r. Such a dependence of the string tension on r appears if the creation of virtual light $q\bar{q}$ pairs is taken into account, causing a flattening of the confining potential at large distances, ≥ 1.0 fm. This effect may become important for bottomonium states with $R(nL) \geq 1.0$ fm, giving a decrease of the masses (e.v.). The explicit expression of $\sigma(r)$ is taken here from [29], where it was deduced from the analysis of radial Regge trajectories of light mesons:

$$\sigma(r) = \sigma_0(1 - \gamma f(r)) \tag{8}$$

with the parameters taken from [29]: $\gamma = 0.40$, $f(r \to 0) = 0$, $f(r \to \infty) = 1.0$.

TABLE 2: The spin-averaged masses $M(nL)$ (MeV) of higher bottomonium states in the NR case, for the SSE, and in einbein approximation (EA) for the potential $V_B(r)$ (5)

State	NR	SSE	EA	Exp. [1]
$2F$	10623	10607	10613	-
$4S$	10657	10630	10645	10579.4±1.2 (4^3S_1)
$3D$	10717	10698	10705	-
$4P$	10808	10783	10795	-
$3F$	10857	10835	10844	-
$5S$	10894	10862	10880	10865±8 ($\Upsilon(10860)$)
$4D$	10942	10916	10928	-
$5P$	11024	10998	11009	-
$6S$	11100	11067	11084	11019±8 ($\Upsilon(11020)$)
$5D$	11139	11109	11123	-
$7S$	11278	11240	11262	-
$6D$	11310	11270	11295	-

In Tables 1 and 2 the masses $M(nL)$ are given only for the flattening potential (8): for low-lying levels they coincide with the masses calculated using the linear potential (with $\sigma = \text{const} = \sigma_0$) within ≤ 2 MeV. For higher states and using the flattening potential, the masses (e.v.) are smaller by $\sim 10 - 60$ MeV (see the numbers in Table 3). In particular, the mass difference is only 12 MeV for the $4S$ and $3D$ states and already 40 MeV for the $6S$ and $5D$ states, reaching 64 MeV for the $7S$ state. It is evident that for a flattening potential the masses $M(nS)$ (any n) are closer to the experimental values.

For a comparison in Tables 1 and 2 the masses $M_{\rm NR}(nL)$, calculated in the NR approximation (where $\omega(nL)$=const=m_b for all states) are also given for the same static potential. These masses $M_{\rm NR}(nL)$ are always

TABLE 3: The masses of higher bottomonium states (MeV) for the static potential Eq. (5) with the parameters (6) and two confining potentials: linear with $\sigma_0 = 0.178$ GeV^2 and the flattening potential Eq. (8)

State	$2D$	$4S$	$3D$	$4P$	$5S$	$4D$	$6S$	$5D$	$7S$
Linear	10456	10656	10717	10812	10901	10950	11122	11163	11326
Flatt.	10450	10645	10705	10795	10880	10928	11084	11123	11262

10−20 MeV larger than in the EA, but in its turn the EA masses are 10−20 MeV larger that the e.v. of the SSE. Such a small difference between the EA and SSE masses is taken into account here by including it in the theoretical error.

For our further analysis it is also important that due to the flattening effect the w.f. at the origin (for higher states) are becoming significantly smaller than for the linear potential, providing a better agreement with the experimental dielectron widths.

Sometimes the point of view is taken that in bottomonium the nonperturbative effects (determined by the confining potential) play an insignificant role for low-lying levels. To clarify this point we have compared two m.e. for a given nS state: of the confining (nonperturbative) potential $< \sigma(r)r >$ and of the gluon-exchange (GE) (or "perturbative") potential, respectively, introducing their ratio $\eta(nS)$:

$$\eta(nS) = \frac{< \sigma(r)r >_{nS}}{< |V_{\rm GE}(r)| >_{nS}}. \tag{9}$$

The results of our calculations are presented in Table 4.

TABLE 4: The ratios $\eta(nS)$

State	$1S$	$2S$	$3S$	$4S$	$5S$	$6S$
$\eta(nS)$	0.24	0.93	1.80	2.78	3.87	5.12

The values of $\eta(nS)$ from Table 4 show that only for the $1S$ ground state the nonperturbative contribution is rather small, equal to 24%, while already for the $2S$ state both contributions are equally important. For higher nS states the nonperturbative contribution dominates, being $\sim (n-1)$ times larger than the perturbative one. For that reason the GE potential can even be considered as a perturbation for higher resonances.

We estimate the accuracy of our calculations to be equal to 15 MeV. The calculated masses weakly depend on the admissible variations of the parameters taken (the same accuracy was obtained in studies of heavy-light mesons [24] and the charmonium family [12]). Still, for higher resonances the accuracy of the calculated masses is worse, since the influence of open channel(s) is not taken into account. Here we can only estimate possible hadronic (decay channel) shifts, while comparing calculated and experimental masses: for $\Upsilon(10580)$ and $\Upsilon(11020)$ a downward shift $\sim 50 \pm 15$ MeV is expected, while the mass $M(5S)$, calculated in single-channel approximation, is close to the experimental mass of $\Upsilon(10860)$ (see Table 2).

Up to now, many bottomonium states, even those which lie below the $B\bar{B}$ threshold, have not yet been discovered, among them the $1D$ multiplet (two states), the $2D$ and $1F$ multiplets, and maybe, the $3P$ multiplet, for which the centroid mass $M(3P) = 10550(15) MeV$[1] , very close to the threshold, is predicted (see Table 1). The observation of these "missing" levels would be very important for the theory.

For further analysis it is also important that the differences between the masses of the $(n+1)S$ and nD states ($n \geq 3$) are small, decreasing for larger n: their values are equal to 60, 48, and 39 MeV for $n = 3$, 4, and 5, respectively (see Table 3).

The w.f. of the nS and nD states are given in the Appendix, together with m.e. like $\omega(nL)$, $\langle \mathbf{p}^2 \rangle$, and those which are needed to determine the dielectron widths and vector decay constants. Also we estimate the relativistic corrections, calculating the velocities v^2/c^2 for different states: their values do not change much, from 0.07 for $\Upsilon(1S)$ up to 0.11 for $\Upsilon(6S)$ (see the Appendix). These numbers illustrate the accuracy of the NR approximation.

[1] Here and below in the brackets we give a theoretical uncertainty

In conclusion we would like to stress two points again: first, in bottomonium the centroid masses $M(nL)$ coincide with the e.v. of the dynamical equation; secondly, the nonperturbative dynamics dominates for all $\Upsilon(nS)$ with $n \geq 2$.

3. DIELECTRON WIDTHS

The dielectron widths are defined here with the help of the van Royen−Weisskopf formula [30] and taking into account the QCD radiative corrections [31]. The widths $\Gamma_{ee}(nS)$ and $\Gamma_{ee}(nD)$ can also be expressed through the vector decay constants f_V, for which explicit expressions were derived in the framework of the field correlator method in [24]. For the S-wave states we have

$$\Gamma_{ee}(n\,^3S_1) = \frac{4\pi e_b^2 \alpha^2}{3M_{nS}} f_V^2(nS)\beta_V = \frac{4e_b^2\alpha^2}{M_{nS}^2}|R_{nS}(0)|^2\xi_{nS}\beta_V, \tag{10}$$

and a similar expression is valid for the D-wave vector states:

$$\Gamma_{ee}(n\,^3D_1) = \frac{4\pi e_b^2 \alpha^2}{3M_{nD}} f_V^2(nD)\beta_V = \frac{4e_b^2\alpha^2}{M_{nD}^2}|R_{nD}(0)|^2\xi_{nD}\beta_V, \tag{11}$$

if the D-wave w.f. at the origin is defined according to the expression Eq. (14) below, which was derived in [32].

In Eqs. (10) and (11) the QCD one-loop perturbative corrections enter via the factor β_V [31]:

$$\beta_V = 1 - \frac{16}{3\pi}\alpha_s(M_V). \tag{12}$$

However, one cannot exclude that higher order perturbative corrections may not be small and therefore, strictly speaking the factor β_V, as well as $\alpha_s(M_V)$ in Eq. (12), has to be considered as an effective constant. Nevertheless this factor cannot be used as an arbitrary parameter. In different approaches its value typically varies in the range 0.75 ± 0.05 [8, 16, 33], which corresponds to an effective coupling $\alpha_s(M_V) = 0.14 \pm 0.04$. (About the choice of the renormalization scale, taken here equal to the mass of a vector $b\bar{b}$ meson M_V, see the discussion in [34]). Here we will neglect in the scale the difference between the mass values for higher states, since all of them lie in narrow range, 10.6−11.1 GeV.

As a first step we analyse here the dielectron widths of low-lying levels $\Gamma_{ee}(\Upsilon(nS))$ $(n = 1, 2, 3)$ and their ratios $r(m/n)$, because these do not depend on the factor β_V. As a second step, the values of β_V are extracted from the magnitudes of the dielectron widths, which are now known with great accuracy owing to the CLEO data [14]. Surprisingly, just the same value $\beta_V = 0.80 \pm 0.01$ is extracted from our fits to three dielectron widths $\Gamma_{ee}(nS)$ $(n = 1, 2, 3)$. This value of β_V corresponds to $\alpha_s(\sim 10.6\,\mathrm{GeV}) = 0.12 \pm 0.01$, which appears to be $\sim 15\%$ smaller than the strong coupling $\alpha_s(10.330\,\mathrm{GeV}) = 0.142 \pm 0.056$, recently extracted from the CLEO data on the total cross sections in e^+e^- annihilation [35].

It is reasonable to assume that such a difference may occur due to second and third order perturbative corrections, which were taken into account in the CLEO analysis, while second and higher-order perturbative corrections to the dielectron widths are not calculated yet. Taking the central value from [35], $\alpha_s(10.330\,\mathrm{GeV}) = 0.142$, one obtains $\beta_V = 0.76$, which is only 5% smaller than our number $\beta_V(M_V) = 0.80$. From this comparison one can estimate that in bottomonium the contribution from unknown higher order corrections to the dielectron width is positive and small, $\leq 10\%$.

In theoretical studies of the dielectron widths and vector decay constants QCD radiative corrections are often neglected, i.e., $\beta_V = 1.0$ is taken [36, 37, 38], while in our analysis only with $\beta_V = 0.80(1)$ a good description of the dielectron widths is achieved. On the contrary, in [17] a significantly smaller number, $\beta_V = 0.46$, is exploited. Probably, such a small value of β_V (or large strong coupling) has been used in [17] in order to suppress the large values of the w.f. at the origin for low-lying states, obtained in their model.

Thus we start with the ratios of the dielectron widths for the $n\,^3S_1$ states $(n = 1, 2, 3)$:

$$r(m/n) = \frac{\Gamma_{ee}(mS)}{\Gamma_{ee}(nS)} = \frac{(M(nS)^2)(R_{mS}(0))^2}{(M(mS)^2)(R_{nS}(0))^2}, \tag{13}$$

which are fully determined by the w.f. at the origin (the masses are known from experiment). Taking the w.f. at the origin calculated here, from the Appendix, one arrives at the values of $r(m/n)$ given in Table 5.

Both the calculated and the experimental ratios agree with each other with an accuracy of $\leq 3\%$ and this result can be considered as a good test of our approach, as well as of the w.f. at the origin calculated here.

TABLE 5: The ratios of the dielectron widths $r(m/n)$ for low-lying $n\,^3S_1$ states

	$r(2/1)$	$r(3/1)$	$r(3/2)$
Theory	0.465	0.339	0.728
Exp. [14]	0.457 ± 0.008	0.329 ± 0.006	0.720 ± 0.016

TABLE 6: The dielectron widths of $n\,^3S_1$ $(n = 1, 2, 3)$ and $n\,^3D_1$ $(n = 1, 2)$ keV with $\beta_V = 0.80$

Widths	Theory	Exp. [14]
$\Gamma_{ee}(1S)$	1.320	1.354±0.024
$\Gamma_{ee}(2S)$	0.614	0.619±0.014
$\Gamma_{ee}(3S)$	0.447	0.446±0.011
$\Gamma_{ee}(1D)$	0.614×10^{-3}	
$\Gamma_{ee}(2D)$	1.103×10^{-3}	

Next we calculate the absolute values of the dielectron widths, which allow to extract the QCD factor β_V. From three dielectron widths $\Gamma_{ee}(nS)$ $(n = 1, 2, 3)$ the same value $\beta_V = 0.80(1)$ has been extracted. Later everywhere $\beta_V = 0.80$ is used, for which the dielectron widths of low-lying and higher states are given in Tables 6 and 7, respectively.

For low-lying levels the dielectron widths (with $\beta_V = 0.80$) agree with the experimental numbers within 3% accuracy (see Table 6).

The dielectron widths calculated here are compared with other theoretical predictions [17, 18] in Table 7: in [17] rather small dielectron widths are obtained, mostly due to the small $\beta_V = 0.46$ taken there. This value is 70% smaller, i.e., the QCD radiative corrections are larger, than in our case. In [18], as well as in our calculations, for $\Upsilon(10580)$ the dielectron width is larger than in experiment, while for the ground state their dielectron width is three times smaller than in our calculations and in experiment.

TABLE 7: The dielectron widths $\Gamma_{ee}(nS)$ (keV) of pure S-wave states

State	$1S$	$2S$	$3S$	$4S$	$5S$	$6S$
GVGV[a] [17]	1.01	0.35	0.25	0.18	0.14	-
CO [18]	0.426	0.356	0.335	0.311	-	-
This paper	1.320	0.614	0.447	0.372	0.314	0.270
Exp. [1]	1.340 ± 0.018	0.612 ± 0.011	0.443 ± 0.008	0.272 ± 0.029	0.31 ± 0.07	0.13 ± 0.03

[a]The numbers given are taken from second paper in [17]

In conclusion of this section we would like to stress again that:

1. The value $\beta_V = 0.80(1)$ should be considered as an effective constant, which implicitly takes into account the contributions from higher perturbative corrections. We expect that higher-order perturbative corrections are positive and rather small, $\leq 10\%$). In the absence of higher corrections, the effective coupling, $\alpha_s(10.6\text{ GeV}) \sim 0.12(1)$ taken here, appears to be slightly smaller than the strong coupling, extracted from the analysis of the cross sections of $e^+e^- \to$ hadrons [35].

2. The calculated ratios of the dielectron widths (for low-lying levels), which are independent of the unknown QCD factor β_V, agree with experimental ratios with an accuracy better than 3%. Therefore one can expect that in our approach the w.f. at the origin (for low-lying levels) are calculated with a good accuracy.

3. The dielectron widths calculated here (with $\beta_V = 0.80$) agree with experiment with an accuracy better than 5%.

4. THE S−D MIXING BETWEEN THE $(n+1)\,^3S_1$ AND $n\,^3D_1$ BOTTOMONIUM STATES

In contrast to the case of the low-lying levels, the calculated dielectron widths of pure nS vector states with $n = 4$ and $n = 6$ exceed the experimental values: for the $4S$ and $6S$ states they are 25% and two times larger than the experimental widths of $\Upsilon(10580)$ and $\Upsilon(11020)$, respectively. Such a suppression of the dielectron widths occurs if one or more channels are open. Some reasons for that have been discussed in [16], where it was that in particular in the Cornell coupled-channel model [39] the dielectron widths of higher charmonium states are not suppressed.

Here as in [13], we assume that an open channel cannot significantly affect the w.f. at the origin calculated in closed-channel approximation. This assumption is based on the study of a four-quark system in [20], where the calculated w.f at the origin of a four-quark system, like $Q\bar{Q}q\bar{q}$, appears to be about two orders smaller than that of a heavy meson $Q\bar{Q}$. We expect this statement also to be true of a continuum w.f. at the origin (the w.f. of an open channel), which can be considered as a particular case of a four-quark system (this does not exclude that a continuum channel can strongly affect the $Q\bar{Q}$ w.f. at large distances). Thus it is assumed that a suppression of the dielectron widths of higher states occurs due to the S—D mixing between the $(n+1)S$ and nD vector states, which happen to have close values of their masses. We also show in the Appendix that in bottomonium the S—D mixing due to tensor forces appears to be very small, giving a mixing angle $\theta_T < 1^\circ$.

For the D-wave states their w. f. at the origin is defined here as in [32]:

$$R_D(0) = \frac{5R''_D(0)}{2\sqrt{2}\omega_b^2}, \tag{14}$$

and for the mixed states their physical (mixed) w.f. are given by

$$R_{\text{phys }S}(0) = \cos\theta R_S(0) - \sin\theta\ R_D(0), \tag{15}$$

$$R_{\text{phys }D}(0) = \sin\theta R_S(0) + \cos\theta\ R_D(0). \tag{16}$$

The w.f. at the origin of pure S- and D-wave states and the derivatives $R''_{nD}(0)$ are given in the Appendix together with other m.e. which are needed to calculate the physical w.f. at the origin (see Table 14). As seen from Table 12, the w.f. $R_{nD}(0)$ are small and therefore the dielectron widths of pure $n\,^3D_1$ states appear to be very small, ≤ 2 eV. Their values are given in Table 8. Notice that our widths are ~ 10 times smaller than those in [36]). On the contrary, the dielectron width of the $4\,^3S_1$ resonance is 25% larger.

To obtain agreement with the experimental value, $\Gamma_{ee}(10580) = 0.273 \pm 0.022$ keV, we take into account the $4S$—$3D$ mixing and determine the mixing angle, $\theta = 27^\circ \pm 4^\circ$ from this fit. Thus $\Upsilon(10580)$ cannot be considered a pure $4S$ vector state, it is mixed with the initially pure $3\,^3D_1$ state. This second "mixed" state will be denoted here as $\tilde{\Upsilon}(\sim 10700)$, it acquires the dielectron width $\Gamma_{ee}(\tilde{\Upsilon}(10700)) = 0.095$ keV, which is 60 times larger than the width of the pure $3\,^3D_1$ state.

The dielectron widths are given in Table 8 in two cases: without the S—D mixing ($\theta = 0$) and for mixed states, taking the same mixing angle $\theta = 27^\circ$ for all higher states.

An interpretation of the experimental width of $\Upsilon(10860)$ cannot be done in an unambiguous way: the calculated $\Gamma_{ee}(5\,^3S_1)$ of a pure $5S$ state ($\theta = 0$) just coincides with the central value of the experimental $\Gamma_{ee}(\Upsilon(10860)) = 0.31 \pm 0.07$ keV. It could mean that for some unknown reason the $5S$ and $4D$ vector states are not mixed. However, there exists another possibility, because the width $\Gamma_{ee}(10860)$ has a rather large experimental error. In particular, for the mixing angle $\theta = 27^\circ$ one obtains $\Gamma_{ee}(\Upsilon(10860)) = 0.23$ keV, which just coincides with the lower bound of the experimental value. To decide which of the two possibilities is realized, more precise measurements of $\Gamma_{ee}(\Upsilon(10860))$ are needed.

An interesting opportunity can be realized for the originally pure $5\,^3D_1$ resonance. The experimental dielectron width of $\Upsilon(11020)$ is very small, $\Gamma_{ee}(11020) = 0.13 \pm 0.03$ keV, i.e., it is two times smaller than the number calculated here without the $6S$—$5D$ mixing (i.e., $\theta = 0$). Even for the mixing angle $\theta = 27^\circ$, the theoretical value is still 26% larger compared to the experimental one (see Table 8). To fit the experimental number a rather large mixing angle, $40^\circ \pm 5^\circ$, has to be taken. For a such a large angle the dielectron widths of both resonances, $\tilde{\Upsilon}(5D)$ (with mass ~ 11120 MeV) and $\Upsilon(11020)$, appear to be almost equal:

$$\begin{cases} \Gamma_{ee}(\Upsilon(11020)) = 0.137 \pm 0.025 \ \text{ keV}, \\ \Gamma_{ee}(\tilde{\Upsilon}(5D)) = 0.135 \pm 0.025 \ \text{ keV}. \end{cases} \tag{17}$$

It is of interest to notice that this large angle is close to the value of the mixing angle $\theta \cong 35^\circ$, which has been extracted in [12] and [40] to fit the dielectron widths in the charmonium family: $\psi(4040)$, $\psi(4160)$, and $\psi(4415)$.

TABLE 8: The dielectron widths (keV) for the mixing angles $\theta = 0$ and $\theta = 27^\circ$ ($\beta_V = 0.80$)

Widths	Theory		Exp. [14]
	no mixing	with mixing	
$\Gamma_{ee}(4S)$	0.372	0.273	0.272±0.029
$\Gamma_{ee}(3D)$	1.435×10^{-3}	0.095	
$\Gamma_{ee}(5S)$	0.314	0.230	0.31±0.07
$\Gamma_{ee}(4D)$	1.697×10^{-3}	0.084	
$\Gamma_{ee}(6S)$	0.270	0.196	0.13±0.03
$\Gamma_{ee}(5D)$	1.878×10^{-3}	0.075	

TABLE 9: The decay constants $f_V(nS)$ (MeV)

State	$1S$	$2S$	$3S$	$4S$	$5S$	$6S$
$\beta_V = 1.0$ [36]	498(20)	366(27)	304(27)	259(22)	228(16)	-
This paper, $\beta_V = 0.80$	794	557	483	383	355	331
f_V(exp) for $\beta_V = 0.80$ [1]	798 ± 6	556 ± 6	481 ± 5	381 ± 19	413 ± 45	268 ± 30
f_V(exp) for $\beta_V = 1.0$ [1]	715 ± 5	497 ± 5	430 ± 4	341 ± 17	369 ± 40	240 ± 27

5. DECAY CONSTANTS IN VECTOR CHANNELS

The decay constant in vector channel $f_V(nL)$ is expressed via the dielectron width in a simple way, as in Eqs. (10) and (11). Therefore, from the experimental widths the "experimental" decay constants can be easily obtained. Still an uncertainty is left, coming from the theoretical error of about 10% in the QCD factor β_V. Also in many papers perturbative one-loop corrections are neglected, i.e., $\beta_V = 1.0$ is taken [36, 37, 38]. This makes a comparison with other calculations more difficult. In our calculations we take $\beta_V = 0.80$, which is slightly more than $\beta_V \sim 0.70$, used in [8] and [33].

To determine the experimental f_V(exp) we take in this section the experimental data from PDG [1] (not the CLEO data [14]), which are used in most of the cited theoretical papers. These decay constants are given in Table 9, both for $\beta_V = 1.0$ and $\beta_V = 0.80$, the difference between them is $\sim 10\%$. The theoretical predictions for f_V give significantly different numbers [36, 37, 38], which shows that the decay constants are rather sensitive to the dynamical parameters of the interaction and the model used. For a comparison we give in Tables 9 and 10 the decay constants $f_V(nS)$ and $f_V(nD)$, calculated here and in Ref. [36], where the relativistic Bethe−Salpeter method was used. All values needed for our calculations are presented in the Appendix.

For a comparison we also mention here the values of $f_V(1S)$ from [37, 38] where the QCD factor $\beta_V = 1.0$ was used: $f_V(1S) = 529$ MeV in [37] is significantly smaller than the value $f_V(1S) = 705(27)$ MeV in [38], which is very close to the "experimental" $f_V(1S)$ (see Table 9). On the contrary, in [33] the perturbative corrections have been taken into account with $\beta_V \sim 0.66$. There the values of f_V are not given but the dielectron widths of the nS vector states ($n = 1, 2, 3$) are in reasonable agreement with experiment.

For the D-wave states the w.f at the origin, the second derivatives, and other m.e. determining the vector decay constants $f_V(nD)$, are given in the Appendix. The calculated $f_V(nD)$ are presented in Table 10 together with the numbers from [36]. Unfortunately, at present there are no experimental data on the dielectron widths for those states.

The decay constants of pure nD vector states ($n = 1, 2, 3$), calculated in our approach, appear to be ~ 10 times smaller than those from [36], where the Bethe−Salpeter equation was used, and the reason behind such a large discrepancy remains unclear. However, if the $4S$—$3D$ mixing is taken into account (with $\theta = 27^\circ$), then the values of $f_V(3D)$ are close to each other in both approaches.

6. SUMMARY AND CONCLUSIONS

In this paper we have calculated the bottomonium spectrum and shown that the masses of the $(n+1)S$ and nD states (for a given $n \geq 2$) are close to each other. We also assume here that between these states S—D mixing

TABLE 10: The decay constants $f_V(nD)$ (MeV)

State	$1D$	$2D$	$3D$	$4D$	$5D$
$\beta_V = 1.0$ [36]	261(21)	155(11)	178(10)	-	-
This paper, $\beta_V = 1.0$, $\theta = 0^\circ$	18	24	28	31	33
This paper, $\beta_V = 1.0$, $\theta = 27^\circ$ [a]	-	-	226	215	206

[a]In bottomonium the $2S-1D$ and $3S-2D$ states, occuring below the threshold, do not mix via tensor forces, see a discussion in Appendix.

takes place, which allows to describe the dielectron widths of higher states with a good accuracy. There are several arguments in favor of such a mixing.

1. Suppression of the dielectron widths of $\Upsilon(10580)$ and $\Upsilon(11020)$.
2. Similarity with the S—D mixing effects in the charmonium family.
3. Strong coupling to the $B\bar{B}$ ($B_s\bar{B}_s$) channel. This fact has been supported by recent observations of the resonances in the processes like $e^+e^- \to \Upsilon(nS)\pi^+\pi^-$ ($n = 1, 2, 3$) and the theoretical analysis in [19].

The important question arises whether it is possible to observe the mixed D-wave vector resonances in e^+e^- experiments. Our calculations give $M(3D) = 10700(15)$ MeV (not including a possible hadronic shift) and $\Gamma_{ee}(\tilde{\Upsilon}(3D)) \sim 95$ eV for the mixing angle $\theta = 27^\circ$, which is three times smaller than $\Gamma_{ee}(\Upsilon(10580))$. For such a width an enhancement of this resonance in the e^+e^- processes might be suppressed, as compared to the peak from the $\Upsilon(10580)$ resonance.

The situation remains unclear with the $5S$—$4D$ mixing, because the dielectron width of $\Upsilon(10860)$ contains a rather large experimental error and a definite conclusion about the value of the mixing angle, or no mixing at all, cannot be drawn. We have considered both cases here, obtaining the mass 10930 ± 15(th) MeV for the $4D$ state.

It looks more probable to observe the resonance $\tilde{\Upsilon}(5D)$ (with the mass ~ 11120 MeV), for which the dielectron width may be almost equal to that of the conventional $\Upsilon(11020)$ resonance. However, since the cross sections of e^+e^- processes depend also on other unknown parameters, like the total width and branching ratio to hadronic channels, the possibility to observe a mixed $5D$ vector resonance may be smaller than for $\Upsilon(11020)$, even for equal dielectron widths.

Recently new observations in the mass region 10.6–11.0 GeV have been reported [41, 42]. The resonance $\Upsilon(10890)$, considered to be identical to $\Upsilon(10860)$, has been observed by the Belle Collaboration [41]. Two resonances in the same region, $\Upsilon(10876)$ and $\Upsilon(10996)$, have been measured by the BaBar Collaboration [42], they are supposed to be the conventional $\Upsilon(10860)$ and $\Upsilon(11020)$. Still there are some differences between the masses and total widths of the resonances from [41, 42], and the PDG data [1], so that further analysis of their parameters is needed. Also one cannot exclude that an overlap of $\Upsilon(11020)$ with the still unobserved $\tilde{\Upsilon}(5D)$ resonance is possible, which can distort the shape and other resonance parameters of the conventional $\Upsilon(11020)$ resonance.

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Appendix
The wave functions at the origin and some matrix elements

We start with the definition of the vector decay constants f_V, for which the following expression was derived in the framework of the vacuum correlator method in [24]:

$$f_V^2(nL) = 12\frac{|\psi_{nS}(0)|^2}{M_V(nL)}\,\xi_V(nL) = \frac{3}{\pi}\frac{|R_{nS}(0)|^2}{M_V(nS)}\,\xi_V(nL). \tag{A.1}$$

Besides the w.f. at the origin, the vector decay constant contains a relativistic factor $\xi_V(nL)$, which was also defined in [24]:

$$\xi_V = \frac{m^2 + \omega^2 + \frac{1}{3}\langle \mathbf{p}^2 \rangle}{2\omega^2}. \tag{A.2}$$

Numerically, $\xi_V(nL)$ is close to unity for all nS and nD vector states. Moreover, for the $(n+1)S$ and nD states, which we are mostly interested in here, their values coincide (see Table 11). Notice that all $\xi_V(nL)$ differ from unity at most by 4%.

TABLE 11: Relativistic factors $\xi_V(nL)$

State	$1S$	$2S$	$3S$	$4S$	$5S$	$6S$	$1D$	$2D$	$3D$	$4D$	$5D$
ξ_V	0.976	0.974	0.972	0.968	0.966	0.963	0.975	0.972	0.968	0.966	0.965

The meaning of the m.e. ω_{nS} and ω_{nD} has been discussed in Section 1 and their values are given in Table 12 together with the m.e. $\langle \boldsymbol{p}^2 \rangle$ and the w.f. at the origin $R_{nS}(0)$ and $R_{nD}(0)$. Notice that again the values of ω_{nL} coincide for $(n+1)S$ and nD states. Knowledge of the m.e. $\langle \mathbf{p}^2 \rangle$ and $\omega(nL)$ allows to define the velocities v^2/c^2 for different states, which in all cases appear to be rather small, ≤ 0.11.

TABLE 12: The matrix elements ω_{nS} (GeV), $\langle \mathbf{p}^2 \rangle$ (GeV^2), and the w.f. at the origin $R_{nS}(0)$ (GeV$^{3/2}$) (no mixing) for the potential (5)

State	$1S$	$2S$	$3S$	$4S$	$5S$	$6S$	$1D$	$2D$	$3D$	$4D$	$5D$
ω	5.02	5.03	5.05	5.08	5.1	5.12	5.02	5.05	5.08	5.1	5.11
$R(0)$	2.529	1.829	1.616	1.506	1.424	1.349	0.0595	0.081	0.0956	0.107	0.1145
$\langle \mathbf{p}^2 \rangle$	1.873	1.907	2.178	2.459	2.689	2.844	1.821	2.136	2.423	2.661	2.810

The w.f. $R_{nD}(0)$ are defined via the derivative $R''_{nD}(0)$ according to the definition (14) from Ref. [32]. Our calculations give the following numbers of the derivatives, presented in Table 13.

TABLE 13: The second derivative $R''_{nD}(0)$ (GeV$^{7/2}$)

state	1D	2D	3D	4D	5D
$R''_{nD}(0)$	0.848	1.167	1.396	1.572	1.692

An interesting feature of the D-wave w.f. is that in contrast to the w.f. $R_{nS}(0)$, which decrease for larger n, the derivatives $R''_{nD}(0)$ and $R_{nD}(0)$ grow for higher radial number n (see Tables 12 and 13), and this effect increases the probability of the S—D mixing.

In bottomonium the w.f. at the origin has been calculated in many papers [16, 18, 43]. Calculated in our approach w.f. for the pure nS and nD states, which are presented in Table 12, allow to determine the physical w.f. at the origin of mixed states for an arbitrary mixing angle. For higher states such physical w.f. at the origin are given in Table 14.

For $\Upsilon(10580)$ and $\Upsilon(10860)$ we take the mixing angle $\theta = 27^\circ$, which was extracted to fit the dielectron width $\Gamma_{ee}(10580)$. For $\Upsilon(11020)$ we take a larger mixing angle, $\theta = 40^\circ$, to fit its dielectron width.

Just the numbers from Table 14 have been used in Sections 4 and 5 to calculate the dielectron widths and the vector decay constants.

In this paper we do not discuss the pseudoscalar decay constants $f_P(nS)$ in bottomonium, although for them the expressions similar to (A.1) have been obtained in [24]; these constants need a special consideration.

In conclusion we also calculate the m.e. over the tensor forces, which are defined by the potential $V_T(r)$:

$$V_T(r) = \frac{4\alpha_s(m_b)}{3\omega_b^2}\frac{1}{r^3}. \tag{A.3}$$

TABLE 14: The physical wave functions at the origin $R_{\rm phys}(0)$ (GeV$^{3/2}$)

Mixing angle	$\theta = 27^\circ$		$\theta = 40^\circ$
State	$\Upsilon(10580)$	$\Upsilon(10860)$	$\Upsilon(11020)$
$R_{\rm phys}(0)$	1.298	1.220	0.960
State	$\tilde\Upsilon(10700)$	$\tilde\Upsilon(10930)$	$\tilde\Upsilon(11120)$
$R_{\rm phys}(0)$	0.769	0.742	0.955

Notice, that in denominator instead of m_b^2, usually used in different models, in the field correlator method the tensor potential (A.3) has to contain the squared dynamical mass $\omega(nL)^2$ [15, 44]. The S—D mixing between $(n+1)S$ and nD vector states, due to tensor interaction, is proportional to the nondiagonal m.e. $\langle r^{-3}\rangle_{(n+1)S,nD}$, for which the following numbers were obtained here (see Table 15).

TABLE 15: The matrix elements $\langle r^{-3}\rangle$ (GeV^3)

n	1	2	3	4	5
$\langle r^{-3}\rangle_{(n+1)S,nD}$	0.0574	0.0683	0.0721	0.0739	0.0744

With the help of these numbers one can easily calculate the m.e. over the tensor potential (A.3), considering it as a perturbation. These m.e. are indeed very small, ≤ 1 MeV and therefore give rise to small contributions to the w.f. at the origin. These small correction to the w.f. at the origin define the mixing angles due to tensor interaction, θ_T, for different states. We obtain that an admissible mixing angle θ_T is small, $\theta_T \leq 1^\circ$.

The vector decay constants were calculated with the use of Eq. (A.1) and presented in Table 16. For higher states their values are given in two cases: without mixing and for $\theta = 27^\circ$. One can see that for the originally pure $4S$ and $6S$ states the constants f_V decrease by 15% and 35%, respectively, compared to the case without mixing. For originally pure D-wave states their decay constants f_V are growing, namely, $f_V(3D)$ and $f_V(5D)$ increase eightfold.

TABLE 16: The decay constants f_V (MeV) for pure and mixed states with $\theta = 27^\circ$

State	$1S$	$2S$	$3S$	$4S$	$5S$	$6S$	$1D$	$2D$	$3D$	$4D$	$5D$
no mixing	794	557	483	444	415	389	18	24	28	31	33
$\theta = 27^\circ$				383	355	331			226	215	206

In Table 16 for low-lying mixed states the values of $f_V(nS)$ (n$=1,2,3$) and $f_V(nD)$ (n$=1,2$) are absent, because these states are indeed pure states, since for them the S—D mixing can occur only via tensor forces, which give very small effect $\theta_T \leq 1^\circ$.

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